Problem A.28

Let

$$\mathsf{T} = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}.$$

(a) Verify that T is hermitian.

(b) Find its eigenvalues (note that they are real).

(c) Find and normalize the eigenvectors (note that they are orthogonal).

(d) Construct the unitary diagonalizing matrix S, and check explicitly that it diagonalizes T.

(e) Check that $\det(\mathsf{T})$ and $\mathrm{Tr}(\mathsf{T})$ are the same for T as they are for its diagonalized form.

Solution

Take the hermitian conjugate of T.

$$\mathsf{T}^\dagger = \widetilde{\mathsf{T}}^* = \begin{pmatrix} 1 & 1+i \\ 1-i & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$$

Since $\mathsf{T}^\dagger = \mathsf{T}$, T is hermitian. As a result, the eigenvalues of T are real, the eigenvectors associated with distinct eigenvalues are orthogonal, and the matrix T is diagonalizable.
Solve the eigenvalue problem for T.

$$\mathsf{Ta} = \lambda\mathsf{a}$$

Bring $\lambda\mathsf{a}$ to the left side and factor a.

$$(\mathsf{T} - \lambda\mathsf{I})\mathsf{a} = \mathsf{0} \tag{1}$$

$\mathsf{a} \neq \mathsf{0}$, so the matrix in parentheses must be singular, that is,

$$\det(\mathsf{T} - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} 1-\lambda & 1-i \\ 1+i & -\lambda \end{vmatrix} = 0.$$

Write out the determinant and solve the equation for λ.

$$(1-\lambda)(-\lambda) - (1+i)(1-i) = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = \{-1, 2\}$$

Let $\lambda_- = -1$ and $\lambda_+ = 2$.

To find the corresponding eigenvectors, plug λ_- and λ_+ back into equation (1).

$$(\mathsf{T}-\lambda_-\mathsf{I})\mathsf{a}_- = 0 \qquad\qquad (\mathsf{T}-\lambda_+\mathsf{I})\mathsf{a}_+ = 0$$

$$\begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} -1 & 1-i \\ 1+i & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2a_1 + (1-i)a_2 = 0 \\ (1+i)a_1 + a_2 = 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} -a_1 + (1-i)a_2 = 0 \\ (1+i)a_1 - 2a_2 = 0 \end{aligned}\right\}$$

$$a_2 = -(1+i)a_1 \qquad\qquad a_1 = (1-i)a_2$$

$$\mathsf{a}_- = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ -(1+i)a_1 \end{pmatrix} \qquad\qquad \mathsf{a}_+ = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (1-i)a_2 \\ a_2 \end{pmatrix}$$

The constants, a_1 and a_2, are arbitrary mathematically due to the fact that the eigenvalue problem is homogeneous. But for the eigenvectors to be physically relevant, a_1 and a_2 need to be chosen so that the magnitude of each eigenvector is one. This is called normalization.

$$|a_1|^2 + |-(1+i)a_1|^2 = 1 \qquad\qquad |(1-i)a_2|^2 + |a_2|^2 = 1$$

$$a_1^2 + (1+i)(1-i)a_1^2 = 1 \qquad\qquad (1-i)(1+i)a_2^2 + a_2^2 = 1$$

$$3a_1^2 = 1 \qquad\qquad 3a_2^2 = 1$$

$$a_1 = \pm\frac{1}{\sqrt{3}} \qquad\qquad a_2 = \pm\frac{1}{\sqrt{3}}$$

Therefore, the normalized eigenvectors corresponding to $\lambda_- = -1$ and $\lambda_+ = 2$ are respectively

$$\mathsf{a}_- = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1+i \end{pmatrix} \quad \text{and} \quad \mathsf{a}_+ = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ 1 \end{pmatrix}.$$

Observe that the eigenvectors are orthogonal.

$$\begin{aligned} \mathsf{a}_-^\dagger \mathsf{a}_+ &= \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 1-i \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \\ &= \frac{1}{3}\left(0\right) \\ &= 0 \end{aligned}$$

In order to diagonalize T, let S^{-1} be the 2×2 matrix whose columns are the eigenvectors of T.

$$\mathsf{S}^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Determine S, the similarity matrix, by finding the inverse of S^{-1}.

$$\left(\begin{array}{cc|cc} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} & 1 & 0 \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} & 1 & 0 \\ 0 & \sqrt{3} & 1+i & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} & 1 & 0 \\ 0 & 1 & \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & i-1 & -\sqrt{3} & 0 \\ 0 & 1 & \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ 0 & 1 & \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{array}\right)$$

Consequently,

$$\mathsf{S} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Note that because the normalized eigenvectors were used for the columns of S^{-1}, this matrix is unitary, and S could have been found more conveniently by taking the hermitian conjugate of S^{-1}. Compute STS^{-1} and verify that T is diagonalizable.

$$\begin{aligned}
\mathsf{STS}^{-1} &= \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2-2i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}
\end{aligned}$$

Calculate the determinant and trace of T.

$$\det(\mathsf{T}) = \begin{vmatrix} 1 & 1-i \\ 1+i & 0 \end{vmatrix} = (1)(0) - (1-i)(1+i) = -2$$

$$\operatorname{Tr}(\mathsf{T}) = \operatorname{Tr} \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} = 1 + 0 = 1$$

Calculate the determinant and trace of STS^{-1}.

$$\det(\mathsf{STS}^{-1}) = \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = (-1)(2) - (0)(0) = -2$$

$$\operatorname{Tr}(\mathsf{STS}^{-1}) = \operatorname{Tr} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = -1 + 2 = 1$$